Inversion Height, Rational Series and Crossed Products

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The Inversion Height of the Free Field is Infinite
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Inversion height of an embedding into a division ring

Let $\iota \colon R \hookrightarrow E$ be an embedding of a domain R into a divison ring E.

Set $E_\iota(0) = R$, and define inductively for $n \geq 0$:

$$E_\iota(n+1) = \begin{array}{c}\text{subring of } E\\ \text{generated by}\end{array} \left\{ r, s^{-1} \mid r, s \in E_\iota(n),\ s \neq 0 \right\}.$$

Then $\bigcup_{n=0}^{\infty} E_\iota(n)$ is the *division ring of fractions of R in E*.

We define $\mathbf{h}_\iota(R)$, the *inversion height of R* (inside E), as ∞ if there is no $n \in \mathbb{N}$ such that $E_\iota(n)$ is a division ring. Otherwise,

$$\mathbf{h}_\iota(R) = \min\{n \mid E_\iota(n) \text{ is a division ring}\}.$$

Examples

- If R is a right or left Ore domain that is not a division ring then the inversion height of $\iota\colon R \to Q_{cl}^{?}(R)$ is 1 and, because of the universal property of the classical ring of quotients, such embedding into a division ring is unique.
- If x and y do not commute and are not invertible in R, then

$$(x^{-1} + (y^{-1} - x)^{-1})^{-1} \in E(3)$$

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but

$$(x^{-1} + (y^{-1} - x)^{-1})^{-1} = x - xyx \in E(0)$$

Free algebra and group algebra over the free group

Let X be a set with, at least two elements, and let k be a (commutative) field.

- (Fisher 71, H -Sánchez 07) proved that there are embeddings into division rings of $k < X >$, where k is a commutative field, of inversion height 1 and 2.
- Similar results hold for the group algebra over the free group over X.

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- Similar results hold for the group algebra over the free group over X.
- In general, if G is an ordered group, Malcev and Neumann showed that kG can be embedded in a “long power series" division ring.

Question (Neumann 49): What is the inversion height of this embedding for kG or for $k < X >$?
It was (re)formulated later by I. Gelfand, S. Gelfand, Retakh and Wilson (2005)

Universal field (=division ring) of fractions

An embedding $\iota\colon R \to E$ with E a divison ring is the universal field of fractions of R if E is rationally generated by R and for any morphism $R \to D$, with D a division ring, there exists a local subring $(T, \mathcal{M})$ of E, containing R, and a morphism $T \to D$ with kernel $\mathcal{M}$ such that the diagram

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & D \\ & T \arrow[ur] & \end{tikzcd}$$

commutes.
Cohn showed that a fir has universal field of fractions (free algebras and the group algebra over a free group are firs...... its universal field of fractions is what is called a free field)

The Malcev-Neumann rational series

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The same is true for the free algebra.
The elements of the universal field of fractions of $R = k < X >$ can be written as

$$(r_1, \dots, r_n) A^{-1} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

where r_i, $s_j \in R$ and A is an $n \times n$ full matrix over R.

Reutenauer’s result

Using that form Reutenauer (1996) showed that: if A is an $n \times n$ matrix whose entries are different elements of X then each entry of its inverse in the universal field of fractions of $k < X >$ has inversion height n.
From this he deduces that the inversion height of the universal field of fractions of $k < X >$ is infinite provided X is an infinite set.

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For example, if $A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$
The 1-1-entry of A^{-1} is $(x_{11} - x_{12}x_{22}^{-1}x_{21})^{-1}$ and has inversion height 2.

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The proof for this result is inspired in an analogous result from Automata Theory and the techniques depend heavily on the commutativity of k

Program for the finite case

The problem for $2 \leq |X| < \infty$ could be solved if one is able do deal with a “well positioned"subalgebra S in which Reutenauer’s Theorem applies

Well positioned means:

- R can be seen as an iterated skew polynomial ring over S
- The universal field of fractions of R can be constructed in three steps:
  -find the universal field of fractions of S, call it E
  -the structure of iterated skew polynomial ring can be extended to one with coefficients in E. Call the resulting Ore domain T.
  -the universal ring of fractions of R is $Q_{cl}(T) = D$

This is going to work because the inversion height of the embedding $S \hookrightarrow E$ is preserved in $R \hookrightarrow D$ (H-Sánchez).

Group crossed products

If R is any ring and G is a group, a crossed product RG is an associative R-ring which is free as a left R-module with basis $\overline{G}$, a copy (as a set) of G.

Multiplication is determined by the rules:

Twisting. For $x, y \in G$, $\bar{x}\bar{y} = \tau(x,y)\overline{xy}$ where $\tau(x,y)$ is a unit of R.
Action. For $x \in G$ and $r \in R$, $\bar{x}r = {}^{\sigma(x)}r\bar{x}$ where ${}^{\sigma(x)}r$ denotes the image of r by an automorphism $\sigma(x)$.

If N is a normal subgroup of G then $RG = RN\frac{G}{N}$, where the latter is some crossed product of the group G/N over the ring RN.

If G/N is finitely generated free abelian, then $RN\frac{G}{N}$ is an iterated skew polynomial ring with only automorphisms and with coefficients in RN.

Lie algebras crossed products

Let L be a Lie k-algebra and let $U(L)$ be its universal enveloping algebra.

Suppose that R is a k-algebra. A *crossed product* $R * U(L)$ such that:

- it has the additive structure of $R \otimes_k U(L)$.
- the product is determined also by an action and a twisting.

If H is an ideal of L, then $R * U(L) = (R * U(H)) * U(L/H)$.

If L/H is finite dimensional abelian Lie algebra, then $(R * U(H)) * U(L/H)$ is an iterated skew polynomial ring with only derivations.

A first solution: the case of free groups

If G is a free group over X, then the results is the literature allow to complete the program we presented before and to give a positive answer to Neumann problem.

Theorem. *Let x, y be two different elements of X. The inverse of an $n \times n$ matrix whose entries are in the set $\{x^i y x^{-i}\}$ have inversion height n in the universal field of fractions of kG.*

Even we can construct examples showing that there are infinitely many non-isomorphic embeddings of kG of infinite inversion height.

A second solution: The case of free Lie algebras

Here the existing theory is not as complete, and we have to prove some analogs of the results for group crossed products in the context of Lie algebras crossed products.

Again we can show combining Reutenauer's result with the reduction via crossed products that the universal field of fractions of $k < X >$ has infinite inversion height.

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What to do next: Can we extend Reutenauer's Theorem to the crossed product situation?

Given $n \in \mathbb{N}$, are there embeddings of the free algebra of inversion height n?